Closing tonight: 3.3
Closing next Fri: 3.4(1), 3.4(2)
Exam 1 is Tuesday in normal quiz section. See website for a reminder of rules and a review sheet/study advice.

3.3 Trig Derivatives (*continued*)

Last time, we showed

$$\lim_{h\to 0}\frac{\sin(x+h)-\sin(x)}{h}=\cos(x)$$

By using a trig identity and the fact

$$\lim_{h\to 0}\frac{\sin(h)}{h}=1.$$

Thus,

$$\frac{d}{dx}(\sin(x))=\cos(x)$$

Similarly, it can be shown that

$$\frac{d}{dx}(\cos(x))=-\sin(x)$$

Entry Task: Use the quotient rule to find the derivatives of

$$1.\ y=\frac{\sin(x)}{\cos(x)}$$

$$2.\ y=\frac{1}{\sin(x)}$$

$\boxed{1}$

$$y'=\frac{(\cos(x))(\cos(x))-(\sin(x))(-\sin(x))}{(\cos(x))^2}$$

$$=\frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)}=\frac{1}{\cos^2(x)}=\sec^2(x)$$

$$\Longrightarrow \frac{d}{dx}(\tan(x))=\sec^2(x)$$

$\boxed{2}$

$$y'=\frac{(\sin(x))(0)-(1)(\cos(x))}{(\sin(x))^2}$$

$$=\frac{-\cos(x)}{\sin^2(x)}=-\frac{1}{\sin(x)}\frac{\cos(x)}{\sin(x)}$$

$$=-\csc(x)\cot(x)$$

$$\frac{d}{dx}(\csc(x))=-\csc(x)\cot(x)$$

$\frac{d}{dx}(\sin(x)) = \cos(x)$	$\frac{d}{dx}(\cos(x)) = -\sin(x)$
$\frac{d}{dx}(\tan(x)) = \sec^2(x)$	$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$
$\frac{d}{dx}(\sec(x)) = \sec(\mathrm{x})\tan(\mathrm{x})$	$\frac{d}{dx}(\csc(x)) = -\csc(\mathrm{x})\cot(\mathrm{x})$

What is the derivative of:

1. $y = x^3 \tan(x)$

$y' = x^3 \sec^2(x) + 3x^2 \tan(x)$

$y' = x^2(x \sec^2(x) + 3\tan(x))$

2. $y = e^x \cos(x) + \frac{3x}{2}$

$y' = e^x \cos(x) + e^x(-\sin(x)) + \frac{3}{2}$

$y' = e^x(\cos(x) - \sin(x)) + \frac{3}{2}$

Side note: You can now use

$$\lim_{h\to 0}\frac{\sin(h)}{h} = 1$$

SAME

Examples: Evaluate

1. $\displaystyle\lim_{x\to 0}\frac{\sin(3x)}{3x} = 1$

OR

LET $t = 3x$

AS $x \to 0$, note that $t \to 0$

$$\lim_{x\to 0}\frac{\sin(3x)}{3x} = \lim_{t\to 0}\frac{\sin(t)}{t} = \boxed{1}$$

2. $\displaystyle\lim_{x\to 0}\frac{\sin(5x)}{4x} =$

$$= \lim_{x\to 0}\frac{5}{4}\,\frac{\sin(5x)}{5x}$$

$$= \frac{5}{4}\lim_{x\to 0}\frac{\sin(5x)}{5x} = \frac{5}{4}\cdot 1 = \boxed{5/4}$$

OR

LET $t = 5x \to x = 1/5\, t$

$$\Rightarrow \lim_{t\to 0}\frac{\sin(t)}{\left(\frac{4}{5}t\right)} = \frac{5}{4}\lim_{t\to 0}\frac{\sin(t)}{t}$$

$$= \boxed{5/4}$$

3. $\displaystyle\lim_{x\to 0}\frac{\sin(6x) - 3e^{x}\sin(6x)}{x} =$

$$= \lim_{x\to 0}\frac{\sin(6x)}{x}\left(1 - 3e^{x}\right)$$

$$= \lim_{x\to 0} 6\,\frac{\sin(6x)}{6x}\left(1 - 3e^{x}\right)$$

$$= 6\cdot 1\cdot\left(1 - 3e^{0}\right) = 6(-2) = \boxed{-12}$$

3.4 Chain Rule

The **composition** of two function is defined by

$$(f \circ g)(x) = f(g(x))$$

Example:

If $f(x) = \sin(x)$, $g(x) = x^3$, then

$$(f \circ g)(x) = f(g(x)) = \sin(x^3).$$

$$x \longrightarrow \underset{\text{"INSIDE"}}{x^3} \longrightarrow \underset{\text{"OUTSIDE"}}{\sin(x^3)}$$

Chain Rule:

$$\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$$

Also written as: $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

Example:

$$\frac{d}{dx}\sin(x^3) = \cos(x^3)\,3x^2$$

$$y = \sin(u) \Rightarrow \frac{dy}{du} = \cos(u)$$

$$u = x^3 \Rightarrow \frac{du}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= \cos(u)\,3x^2$$

$$= \cos(x^3)\,3x^2$$

Here is a brief "proof sketch" for the chain rule:

From the definition of derivative

$$
\begin{aligned}
\frac{d}{dx} f\big(g(x)\big) &= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} \\
&= \lim_{h\to 0} \left(\frac{f(g(x+h)) - f(g(x))}{h} \, \frac{g(x+h)-g(x)}{g(x+h)-g(x)} \right) \\
&= \lim_{h\to 0} \left(\frac{f(g(x+h)) - f(g(x))}{g(x+h)-g(x)} \right) \left(\frac{g(x+h)-g(x)}{h} \right) \\
&= f'\big(g(x)\big) g'(x)
\end{aligned}
$$

Let $j = g(x+h) - g(x)$

$\Rightarrow \quad g(x) + j = g(x+h)$

$$\frac{f(g(x)+j) - f(g(x))}{j}$$

AND AS $h \to 0$, $j \to 0$

$$\lim_{j\to 0} \frac{f(g(x)+j) - f(g(x))}{j} = f'(g(x))$$

Examples: Find the derivative

1. $y = (2x^2 + 1)^2$

$$\left\{ \begin{array}{lll} y = u^2 & \rightarrow & \frac{dy}{du} = 2u \\ u = 2x^2+1 & \rightarrow & \frac{du}{dx} = 4x \end{array} \right.$$

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (2u)\cdot(4x)$$

$$= 2(2x^2+1)\cdot 4x$$

$$y' = 2(2x^2+1)^1 \cdot 4x$$

2. $y = e^{\sin((2x+1)^3)}$

$y = e^u$

$u = \sin(w)$

$w = t^3$

$t = 2x+1$

$$y' = \underbrace{e^{(\sin((2x+1)^3))}}_{e^u} \cdot \underbrace{\cos((2x+1)^3)}_{\cos(w)} \cdot \underbrace{3(2x+1)^2}_{3t^2} \cdot \underbrace{2}_{2}$$

3. $y = \tan(3x + \cos(4x))$

$y = \tan(u)$

$u = 3x + \cos(4x)$

$\frac{d}{dx}(\cos(4x))$

$\frac{dy}{dx} = \underbrace{\sec^2(3x+\cos(4x))} \cdot \underbrace{(3 - \underline{\sin(4x)} \cdot 4)}$

$z = \cos(t)$

$t = 4x$

4. $y = \sin^4(x) = (\sin(x))^4$

$\frac{dy}{dx} = 4(\sin(x))^3 \cdot \cos(x)$

$= 4\sin^3(x)\cos(x)$

5. $y = \sin(x^4)$

$\frac{dy}{dx} = \cos(x^4) \cdot 4x^3$

Identify the "first" rule you would use to differentiate these functions:
(sum, product, quotient or chain?)

1. $y = \sqrt{\sin(x) + x^2 + 1}$ CHAIN

$\boxed{1}$ $y' = \frac{1}{2}(\sin(x)+x^2+1)^{-1/2}(\cos(x)+2x)$

$y' = \dfrac{\cos(x)+2x}{2\sqrt{\sin(x)+x^2+1}}$

2. $y = \dfrac{x^4}{\sin(5x+1)}$ QUOTIENT

$\boxed{2}$ $y' = \dfrac{\sin(5x+1)\cdot 4x^3 - x^4\cos(5x+1)\cdot 5}{\sin^2(5x+1)}$

$y' = \dfrac{x^3(4\sin(5x+1) - 5x\cos(5x+1))}{\sin^2(5x+1)}$

3. $y = \sqrt[3]{4x+1}\cos(\sin(2x))$ PRODUCT

$\boxed{3}$ $y' = (4x+1)^{1/3}(-\sin(\sin(2x))\cos(2x)\cdot 2) + \frac{1}{3}(4x+1)^{-2/3}\cdot 4\cos(\sin(2x))$

4. $y = e^{\tan(x)} - 5(x^8+1)^{50}$ SUM

$\boxed{4}$ $y' = e^{\tan(x)}\sec^2(x) - 250(x^8+1)^{49}\cdot 8x^7$

5. $y = \left(\dfrac{x^2-1}{x^4+1}\right)^{10}$ CHAIN

$\boxed{5}$ $y' = 10\left(\dfrac{x^2-1}{x^4+1}\right)^9 \cdot \dfrac{(x^4+1)2x - (x^2-1)4x^3}{(x^4+1)^2}$